Nature of the Gaussian approximations in phase-ordering kinetics

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The structure of the Gaussian auxiliary field approximation in the theory of phase-ordering kinetics is analyzed with the aim of placing the method within the context of a systematic theory. While we are unable to do this for systems with a scalar order parameter, where the approximation remains uncontrolled, a systematic development about the Gaussian approximation can be outlined for systems with a vector order parameter in terms of a suitably defined $1/N$ expansion.

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I. INTRODUCTION

Although much progress has been made in the understanding of phase-ordering kinetics [1, 2], from the point of view of the theorist, basically, this remains an unsolved problem. The reason is that in the case of a scalar order parameter, which is the most relevant for experiments, no systematic scheme for the development of a perturbation theory is available. This requires the existence of a soluble zero-order approximation which accounts, at least qualitatively, for the relevant physical features of the problem and of a well defined procedure for the calculation, at least in principle, of the higher-order corrections. The case of a system with continous symmetry is in better shape since the $1/N$ expansion meets these requirements, at least in the case of a nonconserved order parameter. For a conserved order parameter there are indications that the large-N limit might be singular [3].

Despite this very unsatisfactory situation, recently much progress has been made in the development of analytical methods for the computation of the structure factor through extensive use of approaches based on the introduction of a Gaussian auxiliary field [4, 5] (GAF), which improve on the original idea of Ohta, Jasnow, and Kawasaki [6] (OJK). An exaustive critical account of these theories has been given by Yeung, Oono, and Shinozaki [5]. The success of this approach, which for the moment is mostly limited to nonconserved order parameter, amounts to the very accurate reproduction of the scaling function for scalar order parameter [4–6] as known from experiments or numerical simulations, and to the prediction of power law tails in the case of vector order parameter [7, 8]. Thus, these theories seem to incorporate those basic ingredients that a real theory of phase-ordering kinetics should have. The shortcoming is that the assumption that the auxiliary field obeys Gaussian statistics is totally uncontrolled. More than theories for the moment these are sophisticated computational prescriptions which are justified *a posteriori*.

A substantial progress toward a systematic theory then would be made if it were possible to identify a scheme within which a GAF approximation plays the role of the zero-order approximation together with the expansion parameter which generates the higher-order corrections. This type of project is illustrated in the paper of Bray and Humayun [9].

Motivated by these considerations here we overview the GAF approximations with the aim of exposing those features which help to put into focus what is required for the eventual development of a systematic theory. For pedagogical reasons we begin with a detailed discussion of the one particle problem which is exactly soluble and therefore allows us to illustrate clearly what is involved in a GAF-type approximation. The same pattern of analysis then will be applied to the field theory case.

II. ONE PARTICLE

Let us consider one particle in a double well potential and in contact with a thermal reservoir. The decay process from the instability point of this system has been thouroughly studied in the literature [10]. In the limit of zero-temperature the equation of motion for the position $\phi(t)$ is given by

$$\dot{\phi} = r\phi - g\phi^3, \tag{2.1}$$

with $r > 0$, $g > 0$. In order to study the quench of this system from high temperature to zero-temperature, let us consider a Gaussian probability distribution for the initial condition $\phi_0 = \phi(t_0)$,

$$P_0(\phi_0) = \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\phi_0^2}{2\Delta}}. \tag{2.2}$$

Due to the symmetry of the problem the average position of the particle vanishes identically $\langle\phi(t)\rangle \equiv 0$ and we concentrate on the behavior of the fluctuations

$$S(t) = \langle\phi^2(t)\rangle = \int_{-\infty}^{+\infty} d\phi P(\phi, t)\phi^2, \tag{2.3}$$

where $P(\phi, t)$ is the probability that the particle occupies the position ϕ at the time t. This quantity can be computed exactly since the equation of motion (2.1) can be solved

$$\phi(t) = f(t - t_0, \phi_0) = \frac{\tau\phi_0}{[1 + (g/r)\phi_0^2(\tau^2 - 1)]^{1/2}}, \tag{2.4}$$

with $\tau = e^{r(t-t_0)}$. Thus, for the probability density we find

$$P(\phi,t) = P_0\big[f^{-1}(t-t_0,\phi)\big]\frac{df^{-1}(t-t_0,\phi)}{d\phi} = \frac{1}{\sqrt{2\pi\Delta\tau^2}}\frac{\exp\{-\frac{1}{2\Delta\tau^2}\frac{\phi^2}{[1-(g/r)\phi^2(1-\tau^{-2})]}\}}{[1-(g/r)\phi^2(1-\tau^{-2})]^{3/2}}, \quad (2.5)$$

where $f^{-1}(t-t_0,\phi)$ is the inverse of $f(t-t_0,\phi_0)$. We insert into (2.3) obtaining

$$S(t) = \frac{r}{g}\frac{\tau^2}{(\tau^2-1)}\{1-\sqrt{\pi x}e^x[1-\mathrm{erf}(\sqrt{x})]\}, \quad (2.6)$$

where $x = r/[2\Delta g(\tau^2-1)]$ and $\mathrm{erf}(z)$ is the error function. For short time (2.6) yields exponential growth of the fluctuations

$$S(t) \sim \Delta\tau^2 \quad (2.7)$$

due to the initial instability, while for large time we get

$$S(t) \sim \frac{r}{g}\frac{\tau^2}{(\tau^2-1)}[1-\sqrt{\pi x}], \quad (2.8)$$

which describes the saturation toward the finite equilibrium value $S(\infty) = \phi_{eq}^2 = r/g$ due to the fact that eventually the particle sits at the bottom of one of the two potential wells and the probability density (2.5) develops two narrow peaks centered about the equilibrium values $\phi_{eq} = \pm\sqrt{r/g}$.

If the exact solution of the problem was not available, this type of behavior could not have been obtained via a straightforward perturbation expansion in the nonlinear coupling g. The zero-order amounts to make the Gaussian approximation in (2.5) which describes only the regime of exponential growth (2.7). Hence, the zero-order theory does not reproduce the qualitative picture of the process, nor is any improvement obtained by taking into account corrections of finite order. The saturation to a finite final equilibrium value is obtained within a perturbative scheme, as shown by Suzuki [10], by resorting to the infinite resummation of the most divergent terms in the series.

However, rather than following this route, let us use the method which in the following will be generalized to the field theory case. The idea is to introduce an auxiliary variable $m(t)$ through a transformation

$$\phi(t) = \sigma(m(t)), \quad (2.9)$$

which takes care of the basic nonlinear features of the problem in such a way that the behavior of $m(t)$ can be treated by straightforward perturbation theory. Namely, the transformation must be such that while $m(t)$ is allowed to grow indefinitely the saturation of $\phi(t)$ to a finite value is induced by σ.

Substituting (2.9) into (2.1) one obtains the equation of motion for $m(t)$

$$\dot{m} = \frac{\sigma(m)}{\sigma'(m)}[r - g\sigma^2(m)], \quad (2.10)$$

with the transformed probability density of initial conditions

$$P_{m_0}(m_0) = P_0(\sigma(m_0)|R)\sigma'(m_0), \quad (2.11)$$

where $m_0 = m(t_0)$ and $P_0(\phi_0|R)$ is the probability density (2.2) conditioned to ϕ_0 belonging to the range of values R for which (2.9) is invertible. Thus, in terms of $m(t)$ we cannot quite get the fluctuations (2.3), but fluctuations conditioned to $\phi \in R$,

$$S(t|R) = \int d\phi P(\phi,t|R)\phi^2 = \int dm P_m(m,t)\sigma^2(m), \quad (2.12)$$

where R is the domain

$$R = (\phi^2 \le r/g) \quad (2.13)$$

and $P_m(m,t)$ is the probability density of m at the time t. How important this restriction is depends on what the statistical weight of trajectories lying outside R is and this in turn is related to the size of the variance Δ of the initial probability density (2.2) compared to the size r/g of the domain R. In the following, we shall neglect the distinction between $S(t|R)$ and $S(t)$ by assuming $\Delta \ll r/g$.

Now, if the transformation σ is such that (2.10) can be solved, at least in perturbation theory, denoting the solution by $m(t) = h(t-t_0,m_0)$ we have

$$P_m(m,t) = P_{m_0}\big[h^{-1}(t-t_0,m)\big]\frac{dh^{-1}(t-t_0,m)}{dm} = P_0(\sigma\big[h^{-1}(t-t_0,m)\big]|R) \times \frac{dh^{-1}(t-t_0,m)}{dm}\sigma'\big[h^{-1}(t-t_0,m)\big], \quad (2.14)$$

which formally solves the problem since it gives an explicit expression for $P_m(m,t)$ in terms of the known quantities P_0,σ, and h.

In order to see how this works in practice, let us go back to the equation of motion (2.10) for $m(t)$ and let us look for the transformation σ which simplifies as much as possible the behavior of $m(t)$. The first attempt is for an outright linearization of (2.10). If this was not possible then, as stated above, σ ought to be such that (2.10) can be solved in perturbation theory. However, in this case, linearization can be achieved. Imposing

$$\frac{\sigma(m)}{\sigma'(m)}[r - g\sigma^2(m)] = rm, \quad (2.15)$$

one finds

$$\phi = \sigma(m) = \frac{m}{[1+(g/r)m^2]^{\frac{1}{2}}} \quad (2.16)$$

and

$$m(t) = h(t-t_0,m_0) = \tau m_0. \quad (2.17)$$

Indeed, we have that while $m(t)$ grows exponentially $\phi(t)$ eventually saturates via (2.16) to the final equilibrium value $\phi_{eq} = \pm\sqrt{r/g}$. Namely, the transformation σ ac-

counts for the nonperturbative features of the problem.

Putting together (2.14), (2.16), and (2.17) we have the exact solution of the problem in terms of the auxiliary variable $m(t)$. The motivation for going to this form of the solution is that in more complicated cases where h and therefore $P_m(m,t)$ cannot be explicitly obtained, the consideration that the auxiliary variable $m(t)$ should not be much affected by the nonlinear nature of the problem authorizes to attempt a Gaussian ansatz for $P_m(m,t)$. This will be the crucial step of the GAF approximation in the phase-ordering problem. The difficulty with an ansatz, however, is that it may not be possible to control the corrections to it. In any case, it should be clear that a Gaussian ansatz does not amount to an overall linearization of the problem, since in (2.12) the ansatz amounts to use a Gaussian form for $P_m(m,t)$ while the nonlinearity remains through the explicit factor $\sigma^2(m)$. To be more specific, since P_0 is Gaussian it is evident from (2.14) that $P_m(m,t)$ is Gaussian if σ and h are linear. Thus, should it be possible to find an expansion parameter λ such that σ and h become linear for $\lambda \to 0$, the Gaussian approximation amounts to take this limit inside $P_m(m,t)$ in (2.12) *but not* in the explicit factor $\sigma^2(m)$.

Let us see how this works in the one particle context. Since in this case $h(t-t_0,m)$ is already linear, in order to make $P_m(m,t)$ Gaussian we need to linearize only σ in (2.14). From (2.16) we may write

$$\sigma^2(\tau^{-1}m) = \tau^{-2}m^2 - \frac{g}{r}\frac{\tau^{-4}m^4}{[1+(g/r)\tau^{-2}m^2]}, \tag{2.18}$$

and using this in (2.14) we obtain

$$P_m(m,t) = P_m^{(0)}(m,t)K(\tau^{-1}m, g/r), \tag{2.19}$$

with

$$P_m^{(0)}(m,t) = \frac{1}{\sqrt{2\pi\Delta\tau^2}}\exp\left\{-\frac{m^2}{2\Delta\tau^2}\right\} \tag{2.20}$$

and

$$K(\tau^{-1}m, g/r) = \frac{\exp\left\{\frac{g}{r}\frac{\tau^{-4}m^4}{[1+(g/r)\tau^{-2}m^2]}\right\}}{[1+(g/r)\tau^{-2}m^2]^{\frac{3}{2}}}. \tag{2.21}$$

Thus, in this case it is possible to identify the nonlinear coupling g with the expansion parameter λ which generates Gaussian statistics for m in the limit $\lambda \to 0$. Then, following the previous discussion, the lowest order is obtained by setting $g \to 0$ in (2.19) but not in the explicit factor $\sigma^2(m)$. From (2.12) we then get

$$S^{(0)}(t) = \int_{-\infty}^{\infty} dm P^{(0)}(m,t)\sigma^2(m)$$
$$= \frac{r}{g}\{1 - \sqrt{\pi y}e^y[1-\mathrm{erf}(\sqrt{y})]\}, \tag{2.22}$$

with $y = r/(2\Delta\tau^2 g)$, which gives $S^{(0)}(t) \sim \Delta\tau^2$ at short time as in (2.7) and $S^{(0)}(t) \sim (r/g)[1-\sqrt{\pi y}]$ for long time. The qualitative effect of the saturation is correctly reproduced, although there is a quantitative discrepancy with (2.8) in the law of approach to equilibrium. In conclusion, in the one particle case the Gaussian approximation can be identified with the zero-order step in a systematic development where higher-order corrections are generated by expanding $K(\tau^{-1}m, g/r)$ in powers of g.

III. PHASE-ORDERING DYNAMICS

Let us now turn to the field theory case. The phase-ordering dynamics following the quench from high temperature to zero-temperature of a system with a nonconserved order parameter is described by the equation of motion

$$\frac{\partial\phi(\vec{x},t)}{\partial t} = \nabla^2\phi(\vec{x},t) - V'(\phi(\vec{x},t)), \tag{3.1}$$

with a Gaussian initial state which generalizes (2.2)

$$P_0[\phi_0(\vec{x})] = \frac{1}{Z_0}e^{-\frac{1}{2\Delta}\int d^dx\phi_0^2(\vec{x})} \tag{3.2}$$

and where $V(\phi)$ is a potential of the double well type.

Again, due to the symmetry of the problem, the average order parameter vanishes identically $\langle\phi(\vec{x},t)\rangle \equiv 0$ and the observable of interest is the equal time correlation function,

$$G(\vec{u},t) = \langle\phi(\vec{x}_1,t)\phi(\vec{x}_2,t)\rangle$$
$$= \int d\phi_1 d\phi_2 P(\phi_1,\vec{x}_1 t;\phi_2,\vec{x}_2 t)\phi_1\phi_2, \tag{3.3}$$

or the structure factor

$$C(\vec{k},t) = \int d^dx e^{i\vec{k}\cdot\vec{u}}G(\vec{u},t), \tag{3.4}$$

where $\vec{u} = \vec{x}_1 - \vec{x}_2$. In (3.3), $P(\phi_1,\vec{x}_1 t;\phi_2,\vec{x}_2 t)$ is the joint probability density that $\phi(\vec{x},t)$ takes the value ϕ_1 at the space-time point $(\vec{x}_1,t)$ and the value ϕ_2 at the space-time point $(\vec{x}_2,t)$.

It has been well established [1, 2], both from experiment and numerical simulations, that in the late stage of the dynamics these quantities obey scaling

$$G(\vec{u},t) \sim f(u/L(t)), \tag{3.5}$$

$$C(\vec{k},t) \sim L^d(t)g(kL(t)), \tag{3.6}$$

where $L(t)$ is the basic length in the problem which is related to the average size of domains and obeys the growth law $L(t) \sim t^{1/2}$, while $f(x)$ and $g(x)$ are scaling functions. The origin of scaling is that in the late stage the order parameter reaches local equilibrium and forms domains of the ordered phases which evolve according to self-similar patterns. From the existence of sharp interfaces separating domains one can deduce [2] the short distance behavior of $f(x)$ or the long wavelength behavior of $g(x)$ (Porod's law)

$$f(u/L) = 1 - 2u/L + \cdots \quad \text{for } u \ll 1, \tag{3.7}$$

$$g(kL) \sim (kL)^{-(d+1)} \quad \text{for } kL \gg 1, \tag{3.8}$$

as well as the saturation law [4] of the order parameter

$$S(t) = G(\vec{u} = 0, t) = \phi_{eq}^2 \left[1 - \frac{a}{L(t)} + O(L^{-2})\right], \tag{3.9}$$

where ϕ_{eq} is the value of the order parameter in the final equilibrium state. Equations from (3.5) to (3.9) contain the minimal phenomenological information that a theory of phase-ordering dynamics should account for.

At this point it is important to emphasize that the scaling behavior described above applies to the late stage of the process where domains are close to saturation and grow through the motion of the interfaces. This stage of the dynamics is dominated by the nonlinear nature of the problem and like in the one particle case it cannot be obtained through any straightforward perturbation expansion. The great difference with the one particle case is that Eq. (3.1) cannot be solved for any realistic potential. Therefore, in order to make analytic progress, we turn to the generalization of the auxiliary variable method.

IV. AUXILIARY FIELD METHOD

Following the idea illustrated above we now introduce an auxiliary field through a local nonlinear transformation

$$\phi(\vec{x}, t) = \sigma(m(\vec{x}, t)), \tag{4.1}$$

which in general is defined through a relation involving the potential

$$K[\sigma(m)] = V'(\sigma). \tag{4.2}$$

We note that such a transformation cannot be a linearizing transformation as it was in the one particle case. In fact, in that case, $m(\vec{x}, t)$ ought to be the free field and the relation between the free field and the interacting field is certainly nonlocal, as it can be easily seen generating the formal solution of (3.1) by iteration. Thus, the transformation (4.1) is introduced in order to take care at least of the gross nonlinear effect which is the saturation of the order parameter to the finite final equilibrium value ϕ_{eq}, leaving the rest, possibly, to perturbation theory. Accordingly, for large time the transformation must go over to the form

$$\sigma(m(\vec{x}, t)) = \phi_{eq}\ \mathrm{sgn}(m(\vec{x}, t)). \tag{4.3}$$

The equation of motion of the auxiliary field is obtained from (3.1)

$$\frac{\partial m}{\partial t} = \nabla^2 m + \frac{1}{\sigma'}[\sigma''(m)(\nabla m)^2 - V'(\sigma)], \tag{4.4}$$

with the transformed initial condition

$$P_{m_0}[m_0(\vec{x})] = P_0[\sigma(m_0(\vec{x}))]J(\phi_0, m_0), \tag{4.5}$$

where $J(\phi_0, m_0)$ is the Jacobian of the transformation (4.1) at the initial time. Representing the solution of (4.4) as a functional of the initial configuration labeled by $\vec{x}$ and t

$$m(\vec{x}, t) = h(\vec{x}, t - t_0; [m_0(\vec{x}')]), \tag{4.6}$$

the probability of a configuration $[m(\vec{x})]$ at the time t can be obtained in terms of the initial probability density (3.2)

$$\begin{aligned} P_m[m(\vec{x}), t] &= P_{m_0}\left[h^{-1}(\vec{x}, t - t_0, [m(\vec{x}')])\right] J(m_0, m) \\ &= P_0\left[\sigma\left(h^{-1}(\vec{x}, t - t_0, [m(\vec{x}')])\right)\right] \\ &\quad \times J(\phi_0, m_0) J(m_0, m), \end{aligned} \tag{4.7}$$

where h^{-1} is the inverse of (4.6) and $J(m_0, m)$ is the Jacobian of this transformation. The above result is the analogue of (2.14) and specifies the statistics of the auxiliary field $m(\vec{x}, t)$ in terms of σ, h and the statistical properties of the initial condition.

Neglecting for simplicity considerations pertaining to the restriction of averages to domains of configurations where (4.1) is invertible, the correlation function (3.3) may be rewritten as

$$G(\vec{u}, t) = \int dm_1 dm_2 P_m(m_1, \vec{x}_1 t; m_2, \vec{x}_2 t)\sigma(m_1)\sigma(m_2), \tag{4.8}$$

where the joint probability of m is related to (4.7) by

$$\begin{aligned} &P_m(m_1, \vec{x}_1 t; m_2, \vec{x}_2 t) \\ &= \int d[m(\vec{x})] P_m[m(\vec{x}), t]\delta(m_1 - m(\vec{x}_1))\delta(m_2 - m(\vec{x}_2)). \end{aligned} \tag{4.9}$$

The above form (4.8) for the correlation function makes a progress over (3.3) if the joint probability of m is available. This requires that the transformation σ is such that Eq. (4.4) for m is soluble. Short of this, as explained in Sec. II, one resorts to the GAF approximation through the linearization of σ and h inside P_0.

Let us now review the predictions of the GAF approximations. If $m(\vec{x}, t)$ is Gaussian, the probability densities are of the form

$$\begin{aligned} &P_m^{(0)}(m_1, \vec{x}_1 t; m_2, \vec{x}_2 t) \\ &= \frac{1}{Z_m} \exp\left\{-\frac{1}{2(1 - \gamma^2)S_0(t)}[m_1^2 + m_2^2 - 2\gamma m_1 m_2]\right\} \end{aligned} \tag{4.10}$$

and

$$P_m^{(0)}(m, \vec{x}t) = \frac{1}{\sqrt{2\pi S_0(t)}} \exp\left\{-\frac{m^2}{2S_0(t)}\right\}, \tag{4.11}$$

with

$$\begin{aligned} &S_0(t) = \langle m^2(\vec{x}, t)\rangle_0 \ , \ G_0(\vec{u}, t) = \langle m(\vec{x}_1, t)m(\vec{x}_2, t)\rangle_0, \\ &\gamma = \gamma(\vec{u}, t) = \frac{G_0(\vec{u}, t)}{S_0(t)} \ , \ Z_m = 2\pi S_0(t)\sqrt{1 - \gamma^2}, \end{aligned} \tag{4.12}$$

and where $\langle\rangle_0$ denotes averages with respect to $P_m^{(0)}$. Hence, for the fluctuations of the order parameter one has

$$S(t) = \int dm P_m^{(0)}(m,\vec{x},t)\sigma^2(m), \tag{4.13}$$

and for the scaling function

$$\begin{aligned} f(u/L(t)) &= \int dm_1 dm_2 P_m^{(0)}(m_1,\vec{x}_1 t; m_2,\vec{x}_2 t) \\ &\quad \times \mathrm{sgn}(m_1)\mathrm{sgn}(m_2) \\ &= \frac{2}{\pi}\sin^{-1}(\gamma). \end{aligned} \tag{4.14}$$

Within this approach the problem is reduced to the computation of $G_0(\vec{u},t)$.

For h to be linear Eq. (4.4) must be of the form

$$\frac{\partial m}{\partial t} = \nabla^2 m + a(t)m, \tag{4.15}$$

where $a(t)$ is some function of time to be determined. Upon linearizing σ the initial probability density (4.5) becomes

$$P_{m_0}[m_0(\vec{x})] = P_0[cm_0(\vec{x})], \tag{4.16}$$

where c is a constant. Solving (4.15) by Fourier transform and averaging over initial conditions with (4.16) one finds

$$C_0(\vec{k},t) = S_0(t)L^d(t)g_0(kL(t)), \tag{4.17}$$

$$G_0(\vec{u},t) = S_0(t)\gamma(|\vec{u}|/L(t)), \tag{4.18}$$

with

$$\begin{cases} L(t) = t^{\frac{1}{2}}, \\ g_0(kL) = \exp\left(-2(kL)^2\right), \\ \gamma(u/L) = \exp(-\frac{u^2}{8L^2}), \\ S_0(t) = \frac{\Delta}{c^2 L^d}\exp\left(2b(t)\right), \\ b(t) = \int_0^t dt' a(t'). \end{cases} \tag{4.19}$$

Inserting the above expression for γ in (4.14) one obtains the Ohta-Jasnow-Kawasaki [6] result for the scaling function which correctly reproduces the behaviors (3.7), (3.8). It is then a matter of studying the behavior of $S(t)$ and for this we must go over to the specific implementations of the method.

A. On site linearization

Making a direct extension to the field theory case of the procedure adopted for one particle, let us look for a transformation σ which linearizes the on site potential in (4.4)

$$-\frac{V'(\sigma)}{\sigma'(m)} = rm, \tag{4.20}$$

where r is a constant. This yields

$$\frac{\partial m}{\partial t} = \nabla^2 m + rm - Q(m)(\nabla m)^2, \tag{4.21}$$

where

$$Q(m) = -\frac{\sigma''(m)}{\sigma'(m)}. \tag{4.22}$$

With the double well potential of the form

$$V(\phi) = -\frac{r}{2}\phi^2 + \frac{g}{4}\phi^4, \tag{4.23}$$

(4.20) reduces to (2.15) yielding as in (2.16)

$$\sigma(m) = \frac{m}{[1+(g/r)m^2]^{\frac{1}{2}}} \tag{4.24}$$

and

$$Q(m) = 3\frac{g}{r}\frac{m}{[1+(g/r)m^2]}. \tag{4.25}$$

However, contrary to what happens for one particle, even though the transformation (4.24) manages to account for the saturation of the order parameter, it is not yet sufficient to linearize the equation of motion. This is done by introducing an approximation, which is optimized by the mean field prescription [11]

$$Q(m)(\nabla m)^2 \to 3\frac{g}{r}\langle(\nabla m)^2\rangle\left\langle\frac{m^2}{1+(g/r)m^2}\right\rangle\frac{m}{\langle m^2\rangle}, \tag{4.26}$$

where averages must be computed self-consistenly. Note that although (4.26) yields the best linear approximation to the equation of motion, it remains an uncontrolled approximation since no small parameter emerges which allows us to compute corrections to it. According to the general discussion made above the implementation the GAF approximation requires, besides the linearization of the equation of motion, also the linearization of σ. Setting $\sigma(m) = m$ in (4.24), the initial condition is given by (4.16) with $c = 1$.

With (4.26) the equation of motion is of the form (4.15) with

$$a(t) = r - 3\frac{g}{r}D_0(t)\frac{S(t)}{S_0(t)}, \tag{4.27}$$

where

$$D_0(t) = \langle(\nabla m)^2\rangle = \int_{\vec{k}} k^2 C_0(\vec{k},t). \tag{4.28}$$

Next, using (4.13),

$$\begin{aligned} S(t) = \frac{r}{g}\Big\{1 - \sqrt{\frac{\pi r}{2gS_0(t)}}e^{\frac{r}{2gS_0(t)}} \\ \times\left[1 - \mathrm{erf}\left(\sqrt{\frac{r}{2gS_0(t)}}\right)\right]\Big\}, \end{aligned} \tag{4.29}$$

and making the assumption to be verified *a posteriori* that $S_0(t)$ grows with time, asymptotically we have

$$S(t) = \frac{r}{g}\Big\{1 - \sqrt{\frac{\pi r}{2gS_0(t)}} + O\Big(\frac{1}{S_0(t)}\Big)\Big\}. \tag{4.30}$$

Inserting into (4.27), to dominant order we get

$$\dot{b}(t) = r - 3\frac{D_0(t)}{S_0(t)} = r + O(t^{-1}), \tag{4.31}$$

which gives $b(t) = rt$. Next, using (4.19) we find

$$S_0(t) = \Delta \frac{\exp\{2rt\}}{t^{\frac{d}{2}}}, \tag{4.32}$$

which is consistent with the assumption made about $S_0(t)$. Finally, inserting the above result into (4.30), we obtain that $S(t)$ saturates exponentially fast to the equilibrium value $\phi_{eq}^2 = r/g$, rather than according to a power law as expected from (3.9).

B. KYG theory

The behavior of $S(t)$ obtained above is what one finds resumming the singular perturbation series of Kawasaki, Yalabik, and Gunton [12] (KYG). The KYG theory is contained in the above treatment as a particular case. If, in addition to the mean field approximation, one makes also an expansion in the nonlinear coupling g, to lowest order $Q(m) \equiv 0$ and the equation of motion becomes

$$\frac{\partial m}{\partial t} = \nabla^2 m + rm, \tag{4.33}$$

namely, the auxiliary field coincides with the free field. The transformation (4.24) together with (4.33) corresponds exactly to the KYG theory, which, therefore, in the present context amounts to the statement that all the important nonlinear features of the problem are adequately taken care of by the transformation (4.24).

C. BH theory

If in (4.26) we keep the first order in g the equation of motion becomes

$$\frac{\partial m}{\partial t} = \nabla^2 m + \left[r - 3\frac{g}{r}\langle(\nabla m)^2\rangle\right] m, \tag{4.34}$$

which is of the type of the equation obtained by Bray and Humayun [9] (BH) starting from an *ad hoc* potential and which leads to the correct behavior for $S(t)$. In fact, in this case (4.27) reduces to

$$\dot{b}(t) = r - 3\frac{g}{r} S_0(t) L^d(t) \int_k k^2 e^{-2k^2 t}, \tag{4.35}$$

and setting to zero the left-hand side for large time

$$S_0(t) \sim L^2(t). \tag{4.36}$$

Inserting this result in (4.30), the behavior (3.9) of $S(t)$ is recovered. This is due to the cancellation of $S_0(t)$ in the denominator of (4.27), which occurs only in first order in g. Notice that from the above result for $S_0(t)$ and the definition (4.19), one obtains $a(t) \sim (d+2)/4t$, which coincides with the form for $a(t)$ introduced by Oono and Puri [13] in their improvement of the OJK theory.

In summary, the GAF approximation obtained via the linearization of the on site potential (i) does not describe correctly the saturation law of the order parameter, except for the very special case where the BH theory applies, and (ii) it is an uncontrolled approximation since there is not a systematic expansion scheme within which it plays the role of the zero-order theory.

D. Mazenko transformation

Let us now go to a different way of introducing the auxiliary field due to Mazenko [4], where equation (4.2) is chosen in such a way that $\sigma(m)$ reproduces the profile of the static interface

$$\sigma''(m) = V'(\sigma). \tag{4.37}$$

In this case, $m(\vec{x}, t)$ has the physical interpretation of the distance to the nearest interface. Using (4.37) in (4.4), we obtain

$$\frac{\partial m}{\partial t} = \nabla^2 m + [1 - (\nabla m)^2] Q(m), \tag{4.38}$$

where $Q(m)$ is still given by (4.22). Note that since $V'(\sigma)$ is an odd function from (4.37) and (4.22), it follows that $Q(m)$ is also an odd function. Thus, the mean field linearization of (4.38) yields

$$\frac{\partial m}{\partial t} = \nabla^2 m + [1 - \langle(\nabla m)^2\rangle] H(t) m, \tag{4.39}$$

where $H(t)$ is some function of time whose explicit form is not important. Hence, Eq. (4.27) now gives

$$\dot{b}(t) = \left[1 - S_0(t) L^d \int_k k^2 e^{-2k^2 t}\right] H(t), \tag{4.40}$$

which, apart for the overall factor $H(t)$, is identical to (4.35) and therefore leads to the same result (4.36) for $S_0(t)$ which yields the correct behavior of $S(t)$. Comparing (4.34) with (4.39), we see that the BH theory is a particular case arising with $H(t)$ constant. Thus, the GAF approximation obtained within the static interface approach yields correct results, but for the same reasons pointed out above it remains an uncontrolled approximation.

E. Generalization of the transformation

We end up this section by considering a generalization of the transformation obtained by allowing for an explicit time dependence. The idea is to see that if in doing so one may get closer to the linearization of the equation for $m(\vec{x}, t)$, as suggested by recent work of Puri and Bray [14]. Replacing (4.1) by

$$\phi(\vec{x}, t) = \sigma(t, m(\vec{x}, t)) \tag{4.41}$$

the equation for m becomes

$$\frac{\partial m}{\partial t} = \nabla^2 m + \frac{1}{\partial\sigma/\partial m}\left[\frac{\partial^2\sigma}{\partial m^2}(\nabla m)^2 - \frac{\partial\sigma}{\partial t} - V'(\sigma)\right]. \tag{4.42}$$

Let us then determine the explicit dependence of σ on t

by imposing

$$\frac{\partial \sigma}{\partial t} = -V'(\sigma), \tag{4.43}$$

which yields

$$\frac{\partial m}{\partial t} = \nabla^2 m - Q(t,m)(\nabla m)^2, \tag{4.44}$$

with

$$Q(t,m) = -\frac{\partial^2 \sigma / \partial m^2}{\partial \sigma / \partial m}. \tag{4.45}$$

Equation (4.43) is nothing but the one particle equation of motion which, using the potential (4.23), yields the solution (2.4), i.e.,

$$\sigma(t,m) = \frac{\tau \sigma(0,m)}{\left[1 + \frac{g}{r}\sigma^2(0,m)(\tau^2 - 1)\right]^{\frac{1}{2}}} \tag{4.46}$$

and

$$Q(t,m) = \frac{\frac{g}{r}\tau^2[\sigma''\sigma^2 - 3\sigma(\sigma')^2] + \{\sigma'' - \frac{g}{r}[\sigma''\sigma^2 - 3\sigma(\sigma')^2)]\}}{\sigma'\left[\frac{g}{r}\sigma^2 - \frac{g}{r}\tau^2\sigma^2 - 1\right]}, \tag{4.47}$$

where the σ's on the right hand side stand for $\sigma(0,m)$ and the primes denote derivatives with respect to m.

Imposing $\sigma''\sigma^2 - 3\sigma(\sigma')^2 = 0$, we find $\sigma(0,m) = \pm(m)^{-1/2}$ and inserting into (4.47) and (4.46) eventually we have

$$\sigma(t,m) = \pm\left[\frac{\tau^2 m}{1 + (g/r)m(\tau^2 - 1)}\right]^{\frac{1}{2}} \tag{4.48}$$

and

$$\frac{\partial m}{\partial t} = \nabla^2 m - \frac{3}{2}\frac{\tau^{-2}}{\left[\tau^{-2}m - \frac{g}{r}(\tau^{-2} - 1)\right]}(\nabla m)^2. \tag{4.49}$$

Indeed, the equation of motion for m is "almost" linear since the nonlinear term vanishes exponentially fast, but the scheme it is not of much use in generating a GAF approximation, since (4.48) cannot be linearized.

V. VECTOR FIELDS

Let us now consider the case of a vector order parameter with N components, $\vec{\phi}(\vec{x}) = [\phi_1(\vec{x}), ..., \phi_N(\vec{x})]$. In this case, a systematic expansion scheme about the GAF approximation can be outlined, although its practical implementation remains to be explored.

Phenomenological expectations in this case are a power law tail in the scaling function of the structure factor [7, 8]

$$g(x) \sim x^{-(d+N)}, \tag{5.1}$$

which generalizes Porod's law and the saturation law [8]

$$S(t) = \phi_{eq}^2\left[1 - \frac{b}{L^2(t)} + O(L^{-3})\right] \tag{5.2}$$

in place of (3.9). Considering the equation of motion

$$\frac{\partial \phi_\alpha(\vec{x},t)}{\partial t} = \nabla^2 \phi_\alpha(\vec{x},t) - \frac{\partial}{\partial \phi_\alpha} V(\vec{\phi}(\vec{x},t)), \tag{5.3}$$

with the potential

$$V(\vec{\phi}) = -\frac{r}{2}\vec{\phi}^2 + \frac{g}{4N}(\vec{\phi}^2)^2 \tag{5.4}$$

the auxiliary field $\vec{m}(\vec{x},t)$ is introduced by generalizing to the vector case the transformation (4.24)

$$\sigma_\alpha(\vec{m}) = \frac{m_\alpha}{\left[1 + \frac{g}{rN}\vec{m}^2\right]^{\frac{1}{2}}}, \tag{5.5}$$

which yields the equation of motion for $\vec{m}$

$$\frac{\partial m_\alpha}{\partial t} = \nabla^2 m_\alpha + r m_\alpha - \frac{g}{rN}\left\{ m_\alpha \sum_\gamma (\nabla m_\gamma)^2 + \frac{\left[2\nabla m_\alpha \cdot \sum_\gamma (m_\gamma \nabla m_\gamma) - \frac{g}{rN} m_\alpha (m_\gamma \nabla m_\gamma)^2\right]}{\left[1 + \frac{g}{rN}\sum_\beta m_\beta^2\right]} \right\}. \tag{5.6}$$

For the equal time correlation function $G(\vec{u},t) = \langle \phi_\alpha(\vec{x}_1,t)\phi_\alpha(\vec{x}_2,t)\rangle$, which is independent of α due to the rotational symmetry of the potential, we have

$$G(\vec{u},t) = \int d\vec{m}_1 d\vec{m}_2 P_m(\vec{m}_1, \vec{x}_1 t; \vec{m}_2, \vec{x}_2 t)\sigma_\alpha(\vec{m}_1)\sigma_\alpha(\vec{m}_2), \tag{5.7}$$

where P_m, which is related to the initial probability density P_0 through the analogues of (4.7) and (4.9), depends explicitly on N through σ and h. As previously stated P_m becomes Gaussian upon linearizing σ and h. We now show that this is achieved by taking the large-N limit. The major difference with the scalar case than is that now there emerges $\lambda = 1/N$ as the natural parameter which yields the Gaussian approximation in the limit $\lambda \to 0$.

Taking the limit $N \to \infty$ terms of the type $\frac{1}{N}\sum_\alpha q_\alpha$ in (5.5) and (5.6) are replaced by the average $\langle q_\alpha \rangle$ yielding the linear equations

$$\sigma_\alpha(\vec{m}) = \frac{m_\alpha}{\left[1 + \frac{g}{r}S_0(t)\right]^{\frac{1}{2}}} \tag{5.8}$$

and

$$\frac{\partial m_\alpha}{\partial t} = \nabla^2 m_\alpha + \left[r - \frac{g}{r}\langle(\nabla m_\alpha)^2\rangle\right] m_\alpha \tag{5.9}$$

since $\langle m_\gamma \vec{\nabla} m_\gamma \rangle$ vanishes. Hence, as anticipated, in the large-N limit the auxiliary field m is Gaussian. Furthermore, Eq. (5.9) is of the BH type yielding (4.36) for $S_0(t) = \langle m_\alpha^2(t)\rangle$.

It is important to realize that the large-N limit we are considering here is quite different from the usual large-N limit [15] performed on the equation of motion (5.3) for $\vec{\phi}$. The latter one is recovered in the present context by taking the large-N limit, namely, using (5.8), also in the explicit σ_α's appearing in (5.7) and eventually obtaining

$$G(\vec{u}, t) = \frac{r}{g}\left[1 - \frac{r}{gL^2}\right] \exp\left(-\frac{u}{8L^2}\right). \tag{5.10}$$

Instead, according to the general structure of the GAF approximation which we have repeatedly illustrated above, N must be kept fixed to whatever value it has been originally specified in the explicit σ_α's in (5.7), while the $N \to \infty$ limit is taken inside P_m. In so doing from (5.5) and (5.7) we obtain the Bray, Puri, and Toyoki [7] (BPT) result for the scaling function

$$f\left(\frac{u}{L(t)}\right) = \langle \hat{m}(\vec{x}_1, t) \cdot \hat{m}(\vec{x}_2, t)\rangle = \frac{N\gamma}{2\pi}\left[B\left(\frac{N+1}{2}, \frac{1}{2}\right)^2 F\left(\frac{1}{2}, \frac{1}{2}; \frac{N+2}{2}, \gamma^2\right)\right], \tag{5.11}$$

where $B(x, y)$ is the β function, $F(a, b; c; z)$ the hypergeometric function, and $\gamma(u/L)$ is given by (4.19). From the above result follows the power law tail (5.1). Furthermore, from $S(t) = \langle \sigma_\alpha^2 \rangle$ we obtain

$$S(t) = \frac{1}{N}\int \frac{d\vec{m}}{(2\pi S_0)^{N/2}} \frac{m^2}{\left[1 + \frac{gm^2}{rN}\right]} e^{-\frac{m^2}{2S_0}}, \tag{5.12}$$

and carrying out the integral

$$S(t) = \frac{Nr}{2g}\left(\frac{Nr}{2gS_0}\right)^{N/2} e^{\frac{Nr}{2gS_0}} \Gamma\left(-\frac{N}{2}, \frac{Nr}{2gS_0}\right), \tag{5.13}$$

where $\Gamma(x, y)$ is the incomplete γ function. Expanding up to first order in $1/S_0$ we obtain

$$S(t) \sim \begin{cases} \phi_{eq}^2\left[1 - \frac{N}{(N-2)}\frac{r}{gS_0(t)}\right] & \text{for } N > 2 \\ \phi_{eq}^2\left[1 + \frac{r}{2gS_0(t)}\right] & \text{for } N = 2, \end{cases} \tag{5.14}$$

which yields the power law behavior (5.2), contrary to the exponential saturation which one obtains in the BTP approach.

These results show that the expected phenomenological behavior is obtained at zero-order within the $1/N$ expansion of the probability density of the auxiliary field. In principle, systematic corrections could be obtained via the higher-order terms in the $1/N$ expansion of P_m, although we do not expect that such a scheme of computation might be easily implemented in practice. It is worth pointing out that the scheme for the systematic improvement of the GAF approximation for vector fields presented here is conceptually different from that proposed by BH in two respects: (i) while we use the standard quartic potential (5.4) BH need to invoke an *ad hoc* potential which cannot even be written in closed form and (ii) the expansion is made in $1/N$, where here N is the number of components of the order parameter rather than the number of components of an additional internal color index. Finally, the comparison between (5.10) on one side and (5.11), (5.14) on the other clearly shows the difference between the standard $1/N$ expansion and the one we have presented here. The most important point is that while there are no localized defects in lowest order in the usual $1/N$ expansion since the correlation function (5.10) decays exponentially, the power law tail (5.1) implied by (5.11) shows that our reformulation of the $1/N$ expansion describes defects in lowest order.

VI. CONCLUDING REMARKS

In conclusion, in this paper, we have analyzed the sequence of steps which must be taken within the framework of a first principles theory in order to generate GAF approximations. The analysis has been restricted to systems with nonconserved order parameter. The idea was to look for the systematic expansion scheme which allows us to control the corrections to the GAF approximations. A project of this type is suggested by the physical motivation behind the introduction of the auxiliary field. This being more smooth and less nonlinear than the order parameter field, hopefully should be tractable in perturbation theory. Our results are negative for the scalar case, in the sense that we are unable to come up with the expansion scheme within which the GAF approximation can be identified with the zero-order approximation. It should be mentioned that there are indications [16,9] that the GAF approximation becomes exact in the limit of infinite space dimensionality, suggesting the $1/d$ expansion as a possible systematic expansion scheme. This is an interesting line of research worth further investigation.

The outlook is somewhat better in the case of a vector order parameter. In this case, one can set up the theory in such a way that the large-N limit yields the GAF approximation. Consequently, one can expect that there exists a $1/N$ expansion where corrections to the GAF approximation are generated systematically. Finally, approximations which go beyond the GAF approximation have been introduced recently by Mazenko [17]. In future work, we plan to look for the connection between that work and the point of view developed here.

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